Lecture 25

Radiation by a Hertzian Dipole

Radiation of electromagnetic field is of ultimate importance for wireless communication systems. The first demonstration of the wave nature of electromagnetic field was by Heinrich Hertz in 1888 [18]. Guglielmo Marconi, after much perserverence with a series of experiments, successfully transmitted wireless radio signal from Cornwall, England to Newfoundland, Canada in 1901 [126]. Hence, radiation by arbitrary sources is an important problem for antennas and wireless communications. We will start with studying the Hertzian dipole which is the simplest of radiation sources we can think of.

25.1 History

The original historic Hertzian dipole experiment is shown in Figure 25.1. It was done in 1887 by Heinrich Hertz [18]. The schematics for the original experiment is also shown in Figure 25.2.

A metallic sphere has a capacitance in closed form with respect to infinity or a ground plane. Hertz could use those knowledge to estimate the capacitance of the sphere, and also, he could estimate the inductance of the leads that are attached to the dipole, and hence, the resonance frequency of his antenna. The large sphere is needed to have a large capacitance, so that current can be driven through the wires. As we shall see, the radiation strength of the dipole is proportional to $p = ql$ the dipole moment.

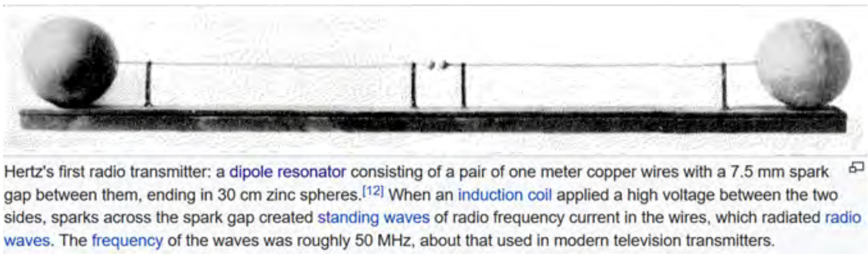

Figure 25.1: Hertz's original experiment on a small dipole (courtesy of Wikipedia [18]).

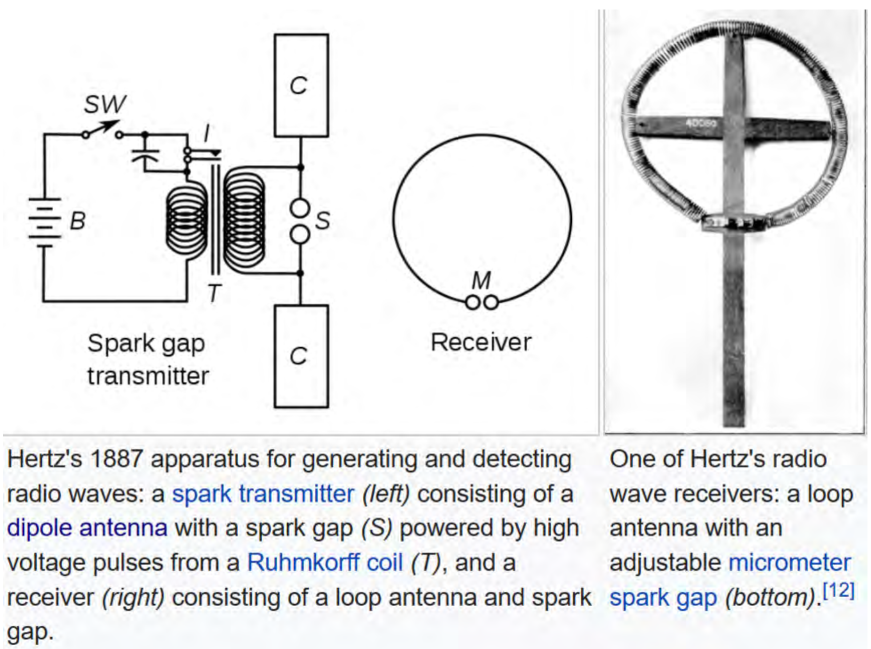

Figure 25.2: More on Hertz's original experiment on a small dipole (courtesy of Wikipedia [18])

25.2 Approximation by a Point Source

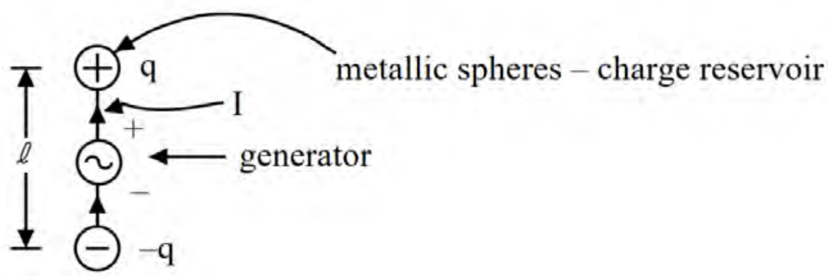

Figure 25.3: Schematics of a small Hertzian dipole.

Figure 25.3 is the schematic of a small Hertzian dipole resembling the original dipole that Hertz made. Assuming that the spheres at the ends store charges of value q, and l is the effective length of the dipole, then the dipole moment $p = ql$. The charge q is varying in time harmonically because it is driven by the generator. Since

$$\frac{dq}{dt} = I,$$

we have

$$Il = \frac{dq}{dt}l = j\omega ql = j\omega p \tag{25.2.1}$$

for this Hertzian dipole.

A Hertzian dipole is a dipole which is much smaller than the wavelength under consideration so that we can approximate it by a point current distribution, or a current density, mathematically given by [31, 39]

$$\mathbf{J}(\mathbf{r}) = \hat{z}Il\delta(x)\delta(y)\delta(z) = \hat{z}Il\delta(\mathbf{r}) \tag{25.2.2}$$

The dipole is as shown in Figure 25.3 schematically. As long as we are not too close to the dipole so that it does not look like a point source anymore, the above is a good mathematical model for describing a Hertzian dipole.

We have learnt previously that the vector potential is related to the current as follows:

$$\mathbf{A}(\mathbf{r}) = \mu \iiint d\mathbf{r}'\mathbf{J}(\mathbf{r}')\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{25.2.3}$$

Since the current is a 3D delta function in space, using the sifting property of a delta function, the corresponding vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r} \tag{25.2.4}$$

Since the vector potential $\mathbf{A}(\mathbf{r})$ is cylindrically symmetric, the corresponding magnetic field is obtained, using cylindrical coordinates, as

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} = \frac{1}{\mu}\left(\hat{\rho}\frac{1}{\rho}\frac{\partial}{\partial \phi}A_z - \hat{\phi}\frac{\partial}{\partial \rho}A_z\right) \tag{25.2.5}$$

where $\frac{\partial}{\partial \phi} = 0$, $r = \sqrt{\rho^2 + z^2}$. In the above, we have used the chain rule that

$$\frac{\partial}{\partial \rho} = \frac{\partial r}{\partial \rho}\frac{\partial}{\partial r} = \frac{\rho}{\sqrt{\rho^2 + z^2}}\frac{\partial}{\partial r} = \frac{\rho}{r}\frac{\partial}{\partial r}.$$

As a result,

$$\mathbf{H} = -\hat{\phi}\frac{\rho}{r}\frac{Il}{4\pi}\left(-\frac{1}{r^2} - j\beta\frac{1}{r}\right)e^{-j\beta r} \tag{25.2.6}$$

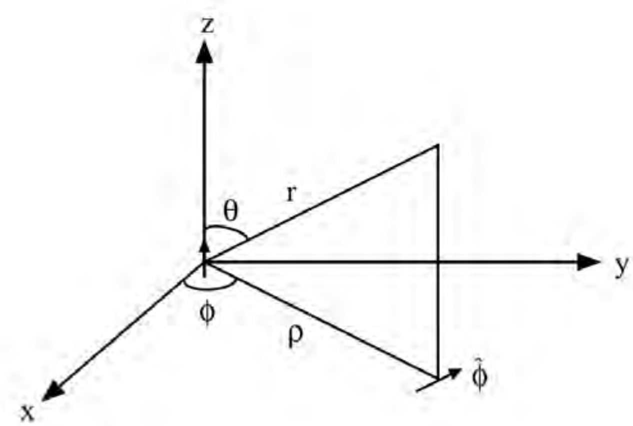

Figure 25.4: Spherical coordinates are used to calculate the fields of a Hertzian dipole.

In spherical coordinates, $\frac{\rho}{r} = \sin\theta$, and (25.2.6) becomes [31]

$$\mathbf{H} = \hat{\phi}\frac{Il}{4\pi r^2}(1 + j\beta r)e^{-j\beta r}\sin\theta \tag{25.2.7}$$

The electric field can be derived using Maxwell's equations.

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H} = \frac{1}{j\omega\epsilon}\left(\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H_\phi - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r}rH_\phi\right) \tag{25.2.8}$$

$$= \frac{Ile^{-j\beta r}}{j\omega\epsilon 4\pi r^3}\left[\hat{r}2\cos\theta(1 + j\beta r) + \hat{\theta}\sin\theta(1 + j\beta r - \beta^2 r^2)\right] \tag{25.2.9}$$

25.2.1 Case I. Near Field, $\beta r \ll 1$

Since $\beta r \ll 1$, retardation effect within this short distance from the point dipole can be ignored. Also, we let $\beta r \to 0$, and keeping the largest terms (or leading order terms in math parlance), then

$$\mathbf{E} \cong \frac{p}{4\pi\epsilon r^3}(\hat{r}2\cos\theta + \hat{\theta}\sin\theta), \qquad \beta r \ll 1 \tag{25.2.10}$$

$$\eta_0\mathbf{H} \ll \mathbf{E}, \qquad \text{when } \beta r \ll 1 \tag{25.2.11}$$

where $p = ql$ is the dipole moment.[1] The above implies that in the near field, the electric field dominates over the magnetic field.

In the above, βr could be made very small by making $\frac{r}{\lambda}$ small or by making $\omega \to 0$. The above is like the static field of a dipole.

Another viewpoint is that in the near field, the field varies rapidly, and space derivatives are much larger than the time derivative.[2]

For instance,

$$\frac{\partial}{\partial x} \gg \frac{\partial}{c\partial t}$$

Alternatively, we can say that the above is equivalent to

$$\frac{\partial}{\partial x} \gg \frac{\omega}{c}$$

or that

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \approx \nabla^2$$

In other words, static theory prevails over dynamic theory. The above approximations are consistent with that the retardation effect is negligible over this lengthscale.

25.2.2 Case II. Far Field (Radiation Field), $\beta r \gg 1$

In this case, retardation effect is important. In other words, phase delay cannot be ignored.

$$\mathbf{E} \cong \hat{\theta} j\omega\mu\frac{Il}{4\pi r}e^{-j\beta r}\sin\theta \tag{25.2.12}$$

and

$$\mathbf{H} \cong \hat{\phi} j\beta\frac{Il}{4\pi r}e^{-j\beta r}\sin\theta \tag{25.2.13}$$

Note that $\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta_0$. Here, $\mathbf{E}$ and $\mathbf{H}$ are orthogonal to each other and are both orthogonal to the direction of propagation, as in the case of a plane wave. Or in a word, a spherical wave resembles a plane wave in the far field approximation.

[1] Here, $\eta_0 = \sqrt{\mu/\varepsilon}$. We multiply $\mathbf{H}$ by η_0 so that the quantities we are comparing have the same unit.

[2] This is in agreement with our observation that electromagnetic fields are great contortionists: They will deform themselves to match the boundary first before satisfying Maxwell's equations. Since the source point is very small, the fields will deform themselves so as to satisfy the boundary conditions near to the source region. If this region is small compared to wavelength, the fields will vary rapidly over a small lengthscale compared to wavelength.

25.3 Radiation, Power, and Directive Gain Patterns

The time average power flow is given by

$$\langle \mathbf{S} \rangle = \frac{1}{2} \Re e[\mathbf{E} \times \mathbf{H}^*] = \hat{r} \frac{1}{2} \eta_0 \left| H_\phi \right|^2 = \hat{r} \frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi r} \right)^2 \sin^2 \theta \tag{25.3.1}$$

The **radiation field pattern** of a Hertzian dipole is the plot of $|\mathbf{E}|$ as a function of θ at a constant $\mathbf{r}$. Hence, it is proportional to $\sin\theta$, and it can be proved that it is a circle.

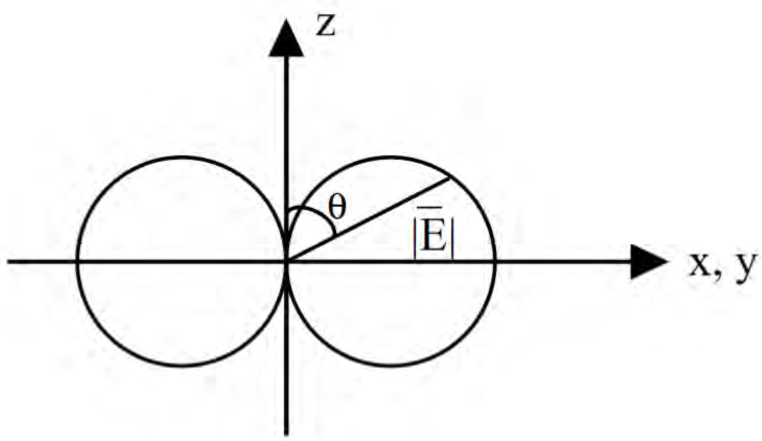

Figure 25.5: Radiation field pattern of a Hertzian dipole. It can be shown that the pattern is a circle.

The **radiation power pattern** is the plot of $\langle S_r \rangle$ at a constant r.

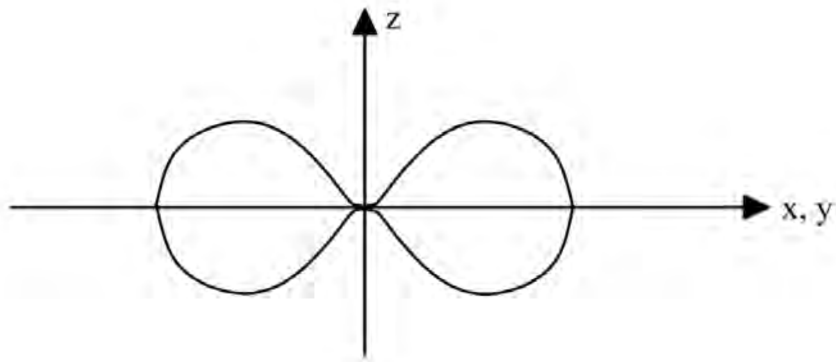

Figure 25.6: Radiation power pattern of a Hertzian dipole which is also the same as the directive gain pattern.

The total power radiated by a Hertzian dipole is given by

$$P = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin\theta \langle S_r \rangle = 2\pi \int_0^{\pi} d\theta \frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi}\right)^2 \sin^3\theta \tag{25.3.2}$$

Since

$$\int_0^{\pi} d\theta \sin^3\theta = -\int_1^{-1} (d\cos\theta)[1-\cos^2\theta] = \int_{-1}^{1} dx(1-x^2) = \frac{4}{3} \tag{25.3.3}$$

then

$$P = \frac{4}{3}\pi\eta_0 \left(\frac{\beta I l}{4\pi}\right)^2 = \frac{\eta_0(\beta I l)^2}{12\pi} \tag{25.3.4}$$

The **directive gain** of an antenna, $G(\theta, \phi)$, is defined as [31]

$$G(\theta,\phi) = \frac{\langle S_r \rangle}{\langle S_{av} \rangle} = \frac{\langle S_r \rangle}{\frac{P}{4\pi r^2}} \tag{25.3.5}$$

where

$$\langle S_{av} \rangle = \frac{P}{4\pi r^2}$$

is the power density if the power P were uniformly distributed over a sphere of radius r. Substituting (25.3.1) and (25.3.4) into the above, we have

$$G(\theta,\phi) = \frac{\frac{\eta_0}{2}\left(\frac{\beta I l}{4\pi r}\right)^2 \sin^2\theta}{\frac{1}{4\pi r^2}\frac{4}{3}\eta_0\pi\left(\frac{\beta I l}{4\pi}\right)^2} = \frac{3}{2}\sin^2\theta \tag{25.3.6}$$

The peak of $G(\theta, \phi)$ is known as the **directivity** of an antenna. It is 1.5 in the case of a Hertzian dipole. If an antenna is radiating isotropically, its directivity is 1. Therefore, the lowest possible values for the directivity of an antenna is 1, whereas it can be over 100 for some antennas like reflector antennas (see Figure 25.7). A **directive gain pattern** is a plot of the above function $G(\theta, \phi)$ and it resembles the radiation power pattern.

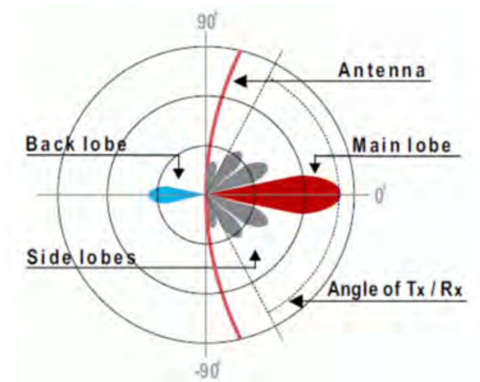

Figure 25.7: The gain of a reflector antenna can be increased by deflecting the power radiated in the desired direction by the use of a reflector (courtesy of racom.eu).

If the total power fed into the antenna instead of the total radiated power is used in the denominator of (25.3.5), the ratio is known as the **power gain** or just **gain**. The total power fed into the antenna is not equal to the total radiated power because there could be some loss in the antenna system like metallic loss.

25.3.1 Radiation Resistance

The **radiation resistance** R_r is the effective resistance that will dissipate the same power as the radiation power P when a current I flows through the resistor. Hence, it is defined by $P = \frac{1}{2} I^2 R_r$, and we have [31]

$$R_r = \frac{2P}{I^2} = \eta_0 \frac{(\beta l)^2}{6\pi} \approx 20(\beta l)^2, \qquad \text{where } \eta_0 = 377 \approx 120\pi \ \Omega \tag{25.3.7}$$

For example, for a Hertzian dipole with $l = 0.1\lambda$, $R_r \approx 8\Omega$.

The above assumes that the current is uniformly distributed over the length of the Hertzian dipole. This is true if there are two charge reservoirs at its two ends. For a small dipole with no charge reservoir at the two ends, the currents have to vanish at the tip of the dipole as shown in Figure 25.8. The effective length of an equivalent Hertzian dipole for the dipole with triangular distribution is **half** of its actual length due to the manner the currents are distributed.[3] Such a formula can be used to estimate the radiation resistance of a dipole.

[3] As shall be shown, when the dipole is short, the details of the current distribution is inessential in determining the radiation field. It is the area under the current distribution that is important.

For example, a half-wave dipole does not have a triangular current distribution a sinusoidal one as shown in Figure 25.9. Nevertheless, we approximate the current distribution of a half-wave dipole with a triangular distribution, and apply the above formula. We pick $a = \frac{\lambda}{2}$, and if we use $l_{\text{eff}} = \frac{\lambda}{4}$ in (25.3.7), we have

$$R_r \approx 50\Omega \tag{25.3.8}$$

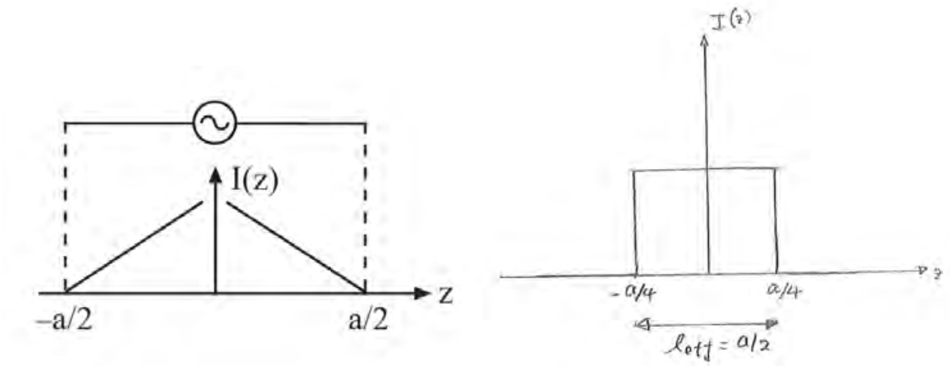

Figure 25.8: The current pattern on a short dipole can be approximated by a triangle since the current has to vanish at the end points of the short dipole. Furthermore, this dipole can be approximated by an effective Hertzian dipole half its length.

The true current distribution on a half-wave dipole resembles that shown in Figure 25.9. The current is zero at the end points, but the current has a more sinusoidal-like distribution like that in a transmission line. Hence, a half-wave dipole is not much smaller than a wavelength and does not qualify to be a Hertzian dipole. Furthermore, the current distribution on the half-wave dipole is not triangular in shape as above. A more precise calculation shows that $R_r = 73\Omega$ for a half-wave dipole [49]. This also implies that a half-wave dipole with sinusoidal current distribution is a better radiator than a dipole with triangular current distribution.

In fact, one can think of a half-wave dipole as a flared, open transmission line. In the beginning, this flared open transmission line came in the form of biconical antennas which are shown in Figure 25.10 [127]. If we recall that the characteristic impedance of a transmission line is $\sqrt{L/C}$, then as the spacing of the two metal pieces becomes bigger, the equivalent characteristic impedance gets bigger. Therefore, the impedance can gradually transform from a small impedance like 50 Ω to that of free space, which is 377 Ω. This impedance matching helps mitigate reflection from the ends of the flared transmission line, and enhances radiation.

Because of the matching nature of bicone antennas, they have a broader bandwidth, and are important in UWB (ultra-wide band) antennas [128].

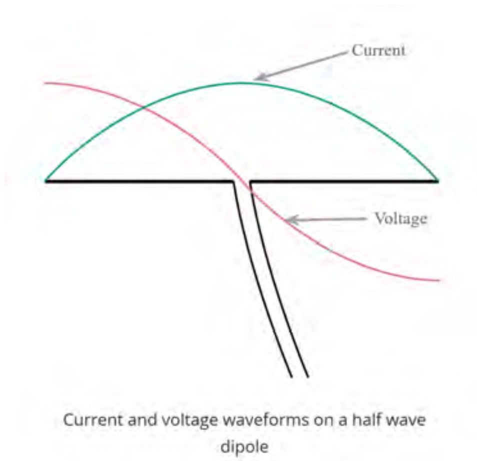

Figure 25.9: A current distribution on a half-wave dipole (courtesy of electronics-notes.co).

Figure 25.10: A bicone antenna can be thought of as a transmission line with gradually changing characteristic impedance. This enhances impedance matching and the radiation of the antenna (courtesy of antennasproduct.com).